

Entanglement entropy after a partial projective measurement in $1 + 1$ dimensional conformal field theories: exact results

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Abstract.

We calculate analytically the Rényi bipartite entanglement entropy S_α of the ground state of $1 + 1$ dimensional conformal field theories (CFT) after performing a projective measurement in a part of the system. We show that the entanglement entropy in this setup is dependent on the central charge and the operator content of the system. When the measurement region A separates the two parts B and \bar{B} , the entanglement entropy between B and \bar{B} decreases like a power-law with respect to the characteristic distance between the two regions with an exponent which is dependent on the rank α of the Rényi entanglement entropy and the smallest scaling dimension present in the system. We check our findings by making numerical calculations on the Klein-Gordon field theory (coupled harmonic oscillators) after fixing the position (partial measurement) of some of the oscillators. We also comment on the post-measurement entanglement entropy in the massive quantum field theories.

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1. Introduction

Entanglement entropy has been playing an important role both in the high-energy and the condensed matter physics for many years. One of the interesting aspects of the bipartite entanglement entropy of the critical systems in $1 + 1$ dimensions is its logarithmic dependence to the subsystem size with a coefficient which is dependent on the central charge of the underlying CFT [1, 2]. Accordingly bipartite entanglement entropy has been used to get information about the underlying quantum field theory of the system. To get more information about the quantum field theory one can either calculate the entanglement entropy of two-disjoint intervals [3, 4] or the entanglement entropy of the excited states of the system [5]. The above quantities are usually dependent on the full operator content of the system and so can fix the CFT. They are also extremely useful in numerical calculations to find the universality class of a fixed point in the studies of quantum phase transitions. Analytical and numerical aspects of bipartite entanglement entropy of $1 + 1$ dimensional critical systems is now well understood. However, little is known about entanglement entropy of tripartite systems. One of the quantities that have been investigated in this regard is the quantum entanglement negativity studied in the context of CFT in [6], see also references therein. It is a very useful quantity when one needs to calculate the bipartite entanglement entropy of two domains after tracing out the third party.

One natural question that arises in the context of tripartite systems regards the behavior of entanglement entropy after partial projective measurement in the system. In other words, consider the ground state of a many body system and make a local projective measurement in a subsystem and then look at the bipartite entanglement entropy of the remaining part. In general, an answer to such kind of questions can be dependent on the basis that one chooses to perform the measurement and of course also to the outcome of the measurement. For example, by considering a simple interacting spin system one can immediately realize that there are infinite possibilities to perform the measurement and so naturally the general solution to such kind of problems looks difficult. However, recently it was possible to show [7] that for quantum chains that can be described by CFT there are some natural bases [8, 9], see also [10, 11, 12] that do not destroy the CFT structure of the system and so one can calculate analytically the bipartite entanglement entropy. When the system under study is a critical system the measurements in the those bases lead to a boundary conformal field theory in the Euclidean space. For this reason, we call them "conformal" bases. However, one should notice that in some cases even if one does the measurement in the mentioned natural bases the outcome of the measurement can be a configuration which from the renormalization group point of view might not flow to a conformal boundary condition, see for example [12]. In [7] it was shown that the ground state after partial measurement still follows the logarithmic law but the formulas change slightly. In this work, we want to generalize the same idea and investigate the post-measurement entanglement entropy in $1 + 1$ dimensional CFTs in different situations. Using the method first introduced

in [1] and further elaborated in [13] we find exact solutions for the post-measurement entanglement entropy in different geometries. Using this method, we will derive some general formulas that the well-known results of the entanglement entropy of a sub-region will be just special cases. We will also find some results regarding the entanglement entropy of disconnected regions. Then we check numerically our findings in the case of the Klein-Gordon field theory by using the projective measurement of a position of the oscillators. We will also provide some numerical results regarding post-measurement entanglement entropy in the massive field theories. It is worth mentioning that similar questions as we are interested here are already discussed in the context of localizable entanglement entropy [14]. In the next section, we first describe the set up of the problem and define our system of interest. Then in section 3, we find exact solutions for the entanglement entropy of the tripartite system. In section 4, we study the finite size effect. In section 5 by numerical means we discuss the post measurement bipartite entanglement entropy in $1 + 1$ dimensional massless and massive Klein-Gordon field theory. Finally in section 6 we summarize our results.

2. Setup and definitions

Consider a quantum system and divide the system into two subsystems D and \bar{D} . The von Neumann entanglement entropy of D with respect to \bar{D} is defined by the following formula:

$$S = -\text{tr } \rho_D \ln \rho_D, \quad (1)$$

where ρ_D is the reduced density matrix of the subsystem D . A simple generalization of the von Neumann entropy which is also a measure of entanglement is called Rényi entropy and is defined as

$$S_\alpha = \frac{1}{1 - \alpha} \ln \text{tr } \rho_D^\alpha. \quad (2)$$

The limit $\alpha \rightarrow 1$ gives back the von Neumann entropy. What we are interested in is the following: consider a generic quantum system in its ground state and make local projective measurements in a sub-region A of this system. After measurement the system collapses to a wave function in which the subsystem A is disentangled from its complement \bar{A} . However, the subsystem \bar{A} has a wave function which is highly entangled. Now divide the subsystem \bar{A} to two subsystems B and \bar{B} in a way that $\bar{A} = B \cup \bar{B}$. What we are interested in is the entanglement entropy between B and \bar{B} . This setup was first studied in [7] for the $1 + 1$ dimensional CFT's when B and \bar{B} are connected and the measurement is done in the conformal basis. Here we study a more general case when A is not necessarily a simply connected domain. The most general case that we would like to study is shown in the Figure 1. It will be soon clear how this setup contains also the cases that have already been discussed in the literature before.

Notation: In this work $s_1(s_2)$ and l represent the characteristic sizes of A and B .

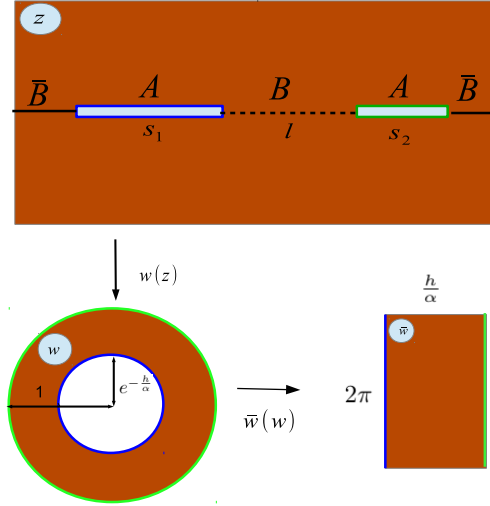


Figure 1. (Color online) Mapping between different regions. The whole plane with two slits A and branch cut (dashed line) on B can be mapped to annulus by the conformal map $w_\alpha(z)$. Then one can go to a cylinder with the conformal map $\bar{w}(w) = \ln w$.

3. Analytical calculation of the post-measurement bipartite entanglement entropy

To calculate the entanglement entropy we generalize the method of [1], see also [13]. It is quite well-known that the Rényi entanglement entropy of a generic system in $d + 1$ dimensions can be written as the ratio of the partition functions in $d + 1$ dimensions as [1, 2]:

$$S_\alpha = \frac{1}{1 - \alpha} \ln \frac{Z_\alpha}{Z_1^\alpha}, \quad (3)$$

where Z_α is the partition function of the system on α -sheeted Riemann surfaces. For example, in two dimensions one just needs to consider the partition function of the system in the presence of a branch cut along the region B that we would like to calculate its entanglement entropy. The above results are valid independent of the boundary conditions and the dimensionality of the system. Consider now the setup that we provided in the previous section. After performing projective measurement in part A of the system that part will be decoupled from the rest, so to calculate the entanglement entropy of B with respect to \bar{B} in the Euclidean language we just need to calculate the partition function of α -sheeted Riemann surfaces with slits on the region A . In other words, the system of interest is as Figure 1 with branch cut on B and slits on A . Different Riemann surfaces are connected through the branch cut. In general calculating the above quantity is a very difficult task, however, when the system is at the critical point and the boundaries are conformally invariant as we will show one can evaluate S_α analytically. As we stated in the introduction the boundary at A is a conformal one if one makes the projective measurement in the conformal bases [8, 7]. For example, for the free bosonic system (coupled harmonic oscillators) the

field ϕ measurement corresponds to Dirichlet boundary condition which is a conformal boundary. It is worth mentioning that since the region A is a bipartite domain in principle one can have different conformal boundary conditions on the two parts of the domain A and this can affect the result of the entanglement entropy. Although this effect does not change the leading term, we will briefly comment about this phenomena later, for a recent discussion see [13].

Instead of directly working with the partition function it is much better to work with the free energy defined as $\mathcal{F}_\alpha = -\ln Z_\alpha$. To calculate the free energy on the Riemann surface with slits we first map the surface to the annulus using the conformal map $w(z)$. As we will show the conformal map that we use to map the Riemann surface to the annulus contributes to the free energy of the Riemann surface. We call this contribution the geometric part of the free energy $\mathcal{F}_\alpha^{geom} = -\ln Z_\alpha^{geom}$, where Z_α^{geom} is the corresponding partition function. After the conformal map one is left with the free energy on the annulus which we call $\mathcal{F}_\alpha^{annn} = -\ln Z_\alpha^{annu}$, where Z_α^{annu} is the corresponding partition function on the annulus. The free energy on the Riemann surface can be calculated explicitly as follows: the change in the free energy of the system after a small horizontal displacement δl of one of the slits is given by the standard result of CFT:

$$\delta \mathcal{F}_\alpha = -\frac{\delta l}{2\pi i} \oint_{\partial S_2} \langle T(z) \rangle dz + c.c., \quad (4)$$

where the integral is along a contour ∂S_2 surrounding one of the slits (here the right one). Under conformal map f the energy momentum tensor transforms according to $T(z) = (\partial_z f)^2 T(f) + (c/12)\{f, z\}$, where $\{f, z\} = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2$ is the Schwarzian derivative. Consequently to calculate the $\delta \mathcal{F}_\alpha$ one needs to map the system to a geometry which the expectation value of the energy-momentum tensor is known. On the cylinder we have $2\langle T(\bar{w}) \rangle = \frac{\delta \mathcal{F}_\alpha^{cyl}}{\delta h_\alpha}$, where h_α is the length of the cylinder. In addition the free energy on the cylinder is related to the free energy on the annulus by the standard formula of CFT, i.e. $\mathcal{F}_\alpha^{cyl} = \mathcal{F}_\alpha^{ann} - \frac{c}{12}h_\alpha$. To calculate the expectation value of the energy-momentum tensor on the Riemann surface we first map the system to the annulus by the conformal map $w_\alpha(z)$ and then to the cylinder by the conformal map $\bar{w}(w) = \ln w_\alpha$. Finally using the above formulas we arrive to the result

$$T(z) = \frac{1}{2}(\partial_z \bar{w}(z))^2 \frac{\delta \mathcal{F}_\alpha^{ann}}{\delta h_\alpha} + \frac{c}{12}\{w_\alpha, z\}. \quad (5)$$

Putting the above formula in (4) and using the fact that $\frac{\delta h_\alpha}{\delta l} = \frac{i}{2\pi} \oint_{\partial S_2} (\partial_z \bar{w})^2$, see for example [15], one arrives to the following result

$$\mathcal{F}_\alpha = \mathcal{F}_\alpha^{geom} + \mathcal{F}_\alpha^{annn} \quad (6)$$

with

$$\frac{\delta \mathcal{F}_\alpha^{geom}}{\delta l} = \frac{ic}{12\pi} \oint_{\partial S_2} \{w_\alpha, z\} dz, \quad (7)$$

The geometric part of the free energy is dependent on only the central charge of the system. However, annulus part of the free energy is dependent on the full operator

content of the CFT [16]. Similar calculations as above are already done in the context of Casimir force [17, 15] and also in the studies of formation probabilities in [18], for other related works see [19]. Note that the free energy of the Riemann surface with two slits is just like the Casimir energy of two slits on the Riemann surface. From this perspective the above calculation provides a connection between post-measurement entanglement entropy and the Casimir energy of two slits.

The conformal map from the plane with two slits on a line with lengths s_1 and s_2 and a branch cut with the length l to annulus with the inner and outer radii $r = e^{-h_\alpha}$ and $r = 1$ with $h_\alpha = \frac{h}{\alpha}$, see Figure 1, has been derived in the appendix. It has the following form:

$$w_\alpha(z) = \left(e^{-\frac{h}{2}} e^{h \frac{\text{sn}^{-1}(\tilde{z}, k^2)}{2\mathcal{K}(k^2)}} \right)^{\frac{1}{\alpha}}, \quad (8)$$

$$h = 2\pi \frac{\mathcal{K}(k^2)}{\mathcal{K}(1 - k^2)}, \quad (9)$$

where \mathcal{K} and sn^{-1} are the elliptic and inverse Jacobi functions \ddagger respectively and

$$\begin{aligned} \tilde{z} &= \frac{2a}{k} \frac{z}{bz + 1} - \frac{1}{k}, \\ a &= \sqrt{\frac{s_2(s_2 + l)}{s_1(s_1 + l)}} \frac{1}{l + s_1 + s_2}, \\ b &= \frac{\sqrt{s_1 s_2 (l + s_1)(l + s_2)} - s_2(l + s_1)}{(l + s_1)(s_1 s_2 - \sqrt{s_1 s_2 (l + s_1)(l + s_2)})}, \end{aligned} \quad (10)$$

with the parameter k given by

$$k = 1 + 2 \frac{s_1 s_2 - \sqrt{s_1 s_2 (l + s_1)(l + s_2)}}{l(l + s_1 + s_2)}. \quad (11)$$

The Schwarzian derivative has poles at $z = 0, s_1, l + s_1$ and $z = s_1 + l + s_2$. Since the contour integral is around the last two poles one can sum over the residue of them and find

$$\frac{\delta \ln Z_\alpha^{\text{geom}}}{\delta l} = \frac{c\alpha}{6} \frac{\left((-2a + b)^2 - b^2 k \right) \left(2\pi^2 - (1 + k(6 + k)) \alpha^2 \mathcal{K}^2(1 - k^2) \right)}{16ak(1 + k) \alpha^2 \mathcal{K}^2(1 - k^2)} \quad (12)$$

Note that an extra α appears because we integrate over α -sheeted surface. Unfortunately, the $\ln Z_\alpha^{\text{geom}}$ can not be calculated analytically in the most general cases and so one needs to rely either on numerical calculations or study it in special cases. We will provide later some analytical results in special limits.

\ddagger Note that in all of the formulas we adopt the Mathematica convention for all the elliptic functions.

The annulus part of the partition function can be written in two equivalent forms as follows[16]:

$$\ln Z_{\alpha}^{annu} = \ln[q_{\alpha}^{-c/24}(1 + \sum_j n_j q_{\alpha}^{\Delta_j})] - c \frac{h}{12\alpha}, \quad (13)$$

$$\ln Z_{\alpha}^{annu} = \ln[\tilde{q}_{\alpha}^{-c/24}(b_0^2 + \sum_j b_j^2 \tilde{q}_{\alpha}^{\Delta_j})] - c \frac{h}{12\alpha}, \quad (14)$$

where n_j and b_j are numbers and Δ_j in the first formula is the boundary scaling dimension and in the second formula is the bulk scaling dimension. Finally q_{α} and \tilde{q}_{α} are defined as

$$q_{\alpha} = e^{-\pi \frac{2\pi\alpha}{h}}, \quad \tilde{q}_{\alpha} = e^{-\frac{2h}{\alpha}}. \quad (15)$$

The first parts of the equations (13) and (14) are equal to $\ln Z_{\alpha}^{cyl}$ which is equal to the logarithm of the partition function on the finite cylinder and the term $c \frac{h}{12\alpha}$ takes us back to the annulus. Having the above formulas one can now calculate the entanglement entropy, however, before going to the details of the calculations it is worth mentioning that one can write the equation (3) with respect to the free energy as

$$S_{\alpha} = \frac{1}{\alpha - 1} (\mathcal{F}_{\alpha}^{ann} + \mathcal{F}_{\alpha}^{geom} - \alpha(\mathcal{F}_1^{ann} + \mathcal{F}_1^{geom})). \quad (16)$$

Having equipped with all the necessary formulas we now apply them to the most interesting cases:

3.1. $s_1 = s_2 = s \ll l$:

This is the familiar case of the entanglement entropy of the subsystem with the length l with respect to the rest and can be studied independent of the projective measurement setup [13]. In this case $k = \frac{l}{2s+l} \rightarrow 1$ and then $h \rightarrow -2 \ln \frac{2s}{l} \rightarrow \infty$. In this limit it is much better to work with $\tilde{q} = (\frac{2s}{l})^{\frac{4}{\alpha}} + \dots$ which is small as far as α is not too big. When $s_1 = s_2 = s$ the geometric contribution can be written as

$$\frac{\delta \ln Z_{\alpha}^{geom}}{\delta l} = \frac{c\alpha}{24} \frac{\pi^2(l+2s)^2 - 2(2l^2 + 4ls + s^2)\alpha^2 \mathcal{K}^2(\frac{4s(l+s)}{(l+2s)^2})}{l(l+s)(l+2s)\alpha^2 \mathcal{K}^2(\frac{4s(l+s)}{(l+2s)^2})}. \quad (17)$$

In the limit $s \ll l$ we can expand the above formula and get

$$\frac{\delta \ln Z_{\alpha}^{geom}}{\delta l} = \frac{c\alpha}{6} \left(\frac{1}{\alpha^2} - 1 \right) \left(\frac{1}{l} - \frac{s}{l^2} + \frac{3s^2}{2l^3} - \frac{5s^3}{2l^4} \right) - \frac{c}{64} \frac{s^4(47 - 48\alpha^2)}{\alpha l^5} + \dots \quad (18)$$

where the dots are subleading terms. The first non-zero term for $\alpha = 1$ starts from the fifth term $\frac{s^4}{l^5}$. Note that the $\alpha = 1$ is related to the partition function in the Casimir effect studies [15, 18]. Using (14) the annulus part can be written as

$$\ln Z_{\alpha}^{annu} = 2 \ln b_0 + \frac{b_1^2}{b_0^2} \left(\frac{2s}{l} \right)^{\frac{4\Delta_1}{\alpha}} + \dots, \quad (19)$$

where Δ_1 is the smallest dimension in the conformal tower. Putting all the terms together we have

$$S_\alpha = \frac{c}{6} \frac{1+\alpha}{\alpha} \ln \frac{l}{s} + 2 \ln b_0 + \frac{b_1^2}{b_0^2} \left(\frac{2s}{l} \right)^{\frac{4\Delta_1}{\alpha}} + \dots \quad (20)$$

The first term is the well-known formula of the entanglement entropy of the sub-region [1, 2]. The second term is the famous Affleck-Ludwig boundary term [21] studied already in the context of entanglement entropy in [2]. The third term and the dots are the subleading terms and are dependent on the scaling dimensions Δ_j and are already studied in [20]. Note that the leading term of the entanglement entropy comes from the geometric part of the partition function and the most important subleading terms come from the annulus part of the partition function.

3.2. $s_2 \ll s_1, l$:

This is the limit which has been discussed already in [7]. In this limit we have

$$h = -\ln \frac{s_2 s_1}{2l(l+s_1)} + \dots, \quad (21)$$

$$\tilde{q} = \left(\frac{s_2 s_1}{2l(l+s_1)} \right)^{2/\alpha} + \dots \quad (22)$$

The equation (12) after expansion is

$$\frac{\delta \ln Z_\alpha^{geom}}{\delta l} = \frac{c(1-\alpha^2)}{12\alpha} \frac{2l+s_1}{l(l+s_1)} + \dots \quad (23)$$

Putting all together we have

$$S_\alpha = \frac{c}{12} \left(\frac{1+\alpha}{\alpha} \right) \ln \frac{l(l+s_1)}{s_2 s_1} + 2 \ln b_0 + \frac{b_1^2}{b_0^2} \left(\frac{s_2 s_1}{2l(l+s_1)} \right)^{2\Delta_1/\alpha} + \dots, \quad (24)$$

where the dots are the subleading terms that some of them are Δ_j dependent. Note that again the leading term comes from the geometric part of the free energy and the most important subleading terms are the result of the annulus part of the free energy. The above equation is the same as the result derived in [7] using twist operator technique.

3.3. $l \ll s_1 = s_2 = s$:

In this case we are interested to the entanglement entropy of two regions that are because of the projective measurement region A completely disconnected and far from each other. In this limit it is much better to work with q which is going to be our small parameter as far as α is not too small. We have

$$h = \frac{\pi^2}{\ln \frac{8s}{l}} + \dots, \quad q = \left(\frac{l}{8s} \right)^{2\alpha} + \dots \quad (25)$$

Using the equations (12) and (13) one can get

$$\ln Z_\alpha^{geom} = \frac{c\alpha}{12} \left(\frac{\ln \frac{l}{a}}{2} + \frac{\pi^2}{\alpha^2 \ln \frac{8s}{l}} \right) + \dots, \quad (26)$$

$$\ln Z_\alpha^{annu} = \frac{c\alpha}{12} \left(\ln \frac{8s}{l} - \frac{\pi^2}{\alpha^2 \ln \frac{8s}{l}} \right) + n_1 \left(\frac{l}{8s} \right)^{2\alpha\Delta_1} + \dots, \quad (27)$$

where we take a as the smallest distance between the two regions which its presence is necessary here to keep the integral convergent. After summing over all the terms we have

$$S_\alpha \asymp \begin{cases} \frac{1}{\alpha-1} \left(\frac{l}{8s}\right)^{2\alpha\Delta_1}, & \alpha < 1 \\ \left(\frac{l}{8s}\right)^{2\Delta_1} \ln \frac{l}{8s}, & \alpha = 1 \\ \frac{\alpha}{\alpha-1} \left(\frac{l}{8s}\right)^{2\Delta_1}, & \alpha > 1, \end{cases} \quad (28)$$

where Δ_1 is the smallest boundary scaling dimension in the spectrum of the system. Note that the first two terms of the equation (27) are canceled out by the first two terms of the equation (26) in a way that the leading term of S_α comes from the annulus part of the free energy. The first interesting aspect of the above result is that the two regions although not connected to each other are highly entangled. We will show in the next section that this is mainly because one can consider the post-measurement wave function as the wave function of a non-local Hamiltonian (we call it from now on corresponding post-measurement Hamiltonian) with power-law decaying couplings. This property makes the setup useful to prepare power-law entangled disconnected many body systems in experimental investigations. The second interesting feature is that for large distances if one increases the size of the subsystem l the entanglement entropy will increase with a power-law like function with respect to the l which again one should trace its reason back to the non-local feature of the corresponding post-measurement Hamiltonian. Finally note that the decaying exponent is α -dependent which makes the above result distinct from the entanglement entropy of two disjoint intervals studies in [3, 4, 22, 23]. Since for $\alpha_1 \leq \alpha_2$ we still have $S_{\alpha_2} \leq S_{\alpha_1}$ the above result is consistent with the general monotonicity properties of Rényi entanglement entropy. However, in most of the previous studies for large distances or large sizes one get $\frac{S_{\alpha_2}}{S_{\alpha_1}} \rightarrow r$ where $r > 0$ is a constant. In our case for $\alpha_1 < 1$ the ratio r is zero for large distances and grows with keeping s fixed and increasing l which means that the behavior of the smallest eigenvalues of the system after projective measurement is very different from the cases without projective measurement. Studying the spectrum of the entanglement Hamiltonian after projective measurement might shed more light on the understanding of distinct behavior in the $\alpha < 1$ regime. It is worth mentioning that if Δ_1 is too big then the leading power-law term can come from the the geometric part of the free energy with an exponent which is independent of α .

3.4. Different boundary conditions:

In this subsection we comment on the possible effect of the different boundary conditions on the two slits. When there are two different conditions on the boundaries of the annulus, i.e. A and B , the equations (13) and (14) have the following more general forms [16]:

$$\ln Z_\alpha^{annu} = \ln[q_\alpha^{-c/24}(1 + \sum_j n_j^{AB} q_\alpha^{\Delta_j})] - c \frac{h}{12\alpha}, \quad (29)$$

$$\ln Z_\alpha^{annu} = \ln[\tilde{q}_\alpha^{-c/24}(b_0^A b_0^B + \sum_j b_j^A b_j^B \tilde{q}_\alpha^{\Delta_j})] - c \frac{h}{12\alpha}, \quad (30)$$

where n_j^{AB} are the non-negative integers and $b_j^A = \langle A|j\rangle\rangle$ and $b_j^B = \langle\langle j|B\rangle$ with $|A(B)\rangle$ and $|j\rangle\rangle$ being Cardy and Ishibashi states respectively. Different coefficients are related to each other with the formula $n_j^{AB} = \sum_{j'} S_j^{j'} b_j^A b_{j'}^B$, where $S_j^{j'}$ is the element of the modular matrix S , for more details see [16]. As it should be clear now the effect of the different boundary conditions on the entanglement entropy is subleading. The first non-zero contribution, for example, in the case of $(s_2 \ll s_1, l)$ can be easily calculated by expanding \tilde{q} as before and then putting the result in (30) and finally in (16). The extra contribution coming from the boundary conditions is

$$S^{LA} = \ln b_0^A + \ln b_0^B \quad (31)$$

This is again the Affleck-Ludwig boundary term [21] studied already in the context of entanglement entropy in [16].

4. Finite size effect

In this subsection we extend the results to a periodic system with finite size. The desired conformal map from an infinite cylinder with two aligned slits to annulus is as follows: First we introduce the conformal map $\tilde{z}(z)$, which takes the system from infinite cylinder with two slits to the whole plane with two symmetric aligned slits on the real line as we discussed in the appendix. The map is as follows:

$$\begin{aligned} \tilde{z} &= \frac{e^{2i\pi \frac{\tilde{z}}{L}} + a_0}{b_1 e^{2i\pi \frac{\tilde{z}}{L}} + b_0}, \\ a_0 &= \frac{e^{2i\pi \frac{s_1}{L}}}{N} \left(1 - k - 2e^{2i\pi \frac{l+s_1}{L}} + (1+k)e^{2i\pi \frac{l}{L}} \right), \\ b_1 &= \frac{-1}{N} \left((1-k)e^{2i\pi \frac{l+s_1}{L}} + 2k - (1+k)e^{2i\pi \frac{s_1}{L}} \right), \\ b_0 &= \frac{e^{2i\pi \frac{s_1}{L}}}{N} \left(1 - k + 2ke^{2i\pi \frac{l+s_1}{L}} - (1+k)e^{2i\pi \frac{l}{L}} \right), \\ N &= -2 - e^{2i\pi \frac{l+s_1}{L}}(-1+k) + e^{2i\pi \frac{s_1}{L}}(1+k), \end{aligned} \quad (32)$$

with the k given by

$$k = 1 + 2 \frac{\sin[\frac{\pi s_1}{L}] \sin[\frac{\pi s_2}{L}] - \sqrt{\sin[\frac{\pi s_1}{L}] \sin[\frac{\pi s_2}{L}] \sin[\frac{\pi(s_1+l)}{L}] \sin[\frac{\pi(s_2+l)}{L}]}}{\sin[\frac{\pi l}{L}] \sin[\frac{\pi(l+s_1+s_2)}{L}]}. \quad (33)$$

Now again we can use the same conformal map as (8) to map the whole plane with two aligned symmetric slits and a branch cut to the annulus. The Schwarzian derivative and the integral (7) over the poles at $z = s_1 + l$ and $z = s_1 + l + s_2$ can be calculated using Mathematica. The final result has the following form:

$$\frac{\delta \ln Z_\alpha^{geom}}{\delta l} = -i\pi c \frac{P - \alpha^2 Q \mathcal{K}^2 (1 - k^2)}{\alpha R \mathcal{K}^2 (1 - k^2)} \quad (34)$$

with

$$\begin{aligned}
 P &= 2\pi^2 \left(-4k(e^{2\pi i \frac{l+s_1}{L}} - 1) + (1+k)^2 e^{2\pi i \frac{s_1}{L}} (e^{2\pi i \frac{l}{L}} - 1)^2 \right), \\
 Q &= (1+6k+k^2) \times \\
 &\quad \left(-2(k-1)^2 e^{2\pi i \frac{l+s_1}{L}} - 4k - 4k e^{4\pi i \frac{l+s_1}{L}} + (1+k)^2 e^{2\pi i \frac{s_1}{L}} + (1+k)^2 e^{2\pi i \frac{2l+s_1}{L}} \right), \\
 R &= 48Lk(1+k)^2 (-1 + e^{\frac{2i\pi l}{L}}) (-1 + e^{\frac{2i\pi s_1}{L}}) (-1 + e^{\frac{2i\pi(s_1+l)}{L}}).
 \end{aligned}$$

In the limit $L \rightarrow \infty$ we are back again to the result (12). In the following subsections we will study some special cases of the above formula.

4.1. $s_1 = s_2 = s \ll l$:

This is the familiar case of the entanglement entropy of a subsystem without projective measurement. As before the annulus part does not contribute to the leading term. The derivative of the geometric part of the partition function has the following form

$$\frac{\delta \ln Z_{\alpha}^{geom}}{\delta l} = \frac{\pi(1-\alpha^2)}{6\alpha} \frac{\cot[\frac{\pi l}{L}]}{L} - \frac{\pi^2(1-\alpha^2)}{6\alpha} \frac{s}{L^2 \sin^2[\frac{\pi l}{L}]} + \dots, \quad (35)$$

Where the dots are subleading terms. In this case the annulus part of the free energy does not have any contribution in the leading order. After integration of the equation (35) one can easily find

$$S_{\alpha} = \frac{1}{6} \frac{1+\alpha}{\alpha} \ln \left(\frac{L}{\pi s} \sin \left[\frac{\pi l}{L} \right] \right) + \dots, \quad (36)$$

which is the well-known formula of the entanglement entropy of the subregion [2].

4.2. $s_2 \ll s_1, l$:

In this case as far as α is not too big the small parameter is again \tilde{q} and the annulus part does not play any role in the leading level. After expansion of the formula (34) with respect to s_2 and we have

$$\frac{\delta \ln Z_{\alpha}^{geom}}{\delta l} = \frac{c(1-\alpha^2)}{12\alpha} \frac{\pi \sin[\frac{\pi(2l+s_1)}{L}]}{L \sin[\frac{\pi l}{L}] \sin[\frac{\pi(l+s_1)}{L}]} + \dots \quad (37)$$

The entanglement entropy now can be calculated easily after integration of the above formula. the leading term has the following form:

$$S_{\alpha} = \frac{c}{12} \left(1 + \frac{1}{\alpha} \right) \ln \left(\frac{L \sin \frac{\pi}{L} (l+s_1) \sin \frac{\pi}{L} l}{\pi s_2 \sin \frac{\pi}{L} s_1} \right) + \dots, \quad (38)$$

which is the result of [7]. Note that again the annulus part does not have any contribution in the leading term.

4.3. $l \ll s_2 = s_1 = s$:

In this limit as far as α is not too small we have

$$h = \frac{-\pi^2}{2 \ln \frac{\pi l}{4L}} + \dots, \quad q = \left(\frac{\pi l}{4L}\right)^{4\alpha} + \dots \quad (39)$$

Then after using the equations (13) and (34) we have

$$S_\alpha = \begin{cases} \frac{1}{\alpha-1} \left(\frac{\pi l}{4L}\right)^{4\alpha\Delta_1}, & \alpha < 1 \\ \left(\frac{\pi l}{4L}\right)^{4\Delta_1} \ln \frac{\pi l}{4L}, & \alpha = 1 \\ \frac{\alpha}{\alpha-1} \left(\frac{\pi l}{4L}\right)^{4\Delta_1}, & \alpha > 1, \end{cases} \quad (40)$$

Two comments are in order here. First similar to the derivation of the equation (28) the leading term comes from the annulus part of the free energy. Second the exponents here are two times bigger than the ones we discussed before in the equation (28).

5. Numerical results: Klein-Gordon free field theory

In this section, we verify the exact formulas derived in the last section by performing numerical calculations on the free bosonic system. The Klein-Gordon field theory is defined by the following action:

$$\frac{1}{2} \int \{(\partial\phi(x))^2 + m^2\phi^2(x)\} dx. \quad (41)$$

The procedure that we follow is based on a well-known technique. We first discretize the field theory and get some coupled harmonic oscillators and then use the correlation matrices as first discovered in [24], see also [25] and [26], to calculate the entanglement entropy after projective measurement. The measurement that we are interested in is the field $\phi(x)$ itself which is the most natural basis one can start with. Since fixing $\phi(x)$ is just a Dirichlet boundary condition if one works with a massless system the conformal structure will be still preserved. The upcoming results are also valid if one does the measurements in the momentum basis (related to Neumann boundary condition) but not a generic basis that diagonalizes an arbitrary field.

Consider the Hamiltonian of L -coupled harmonic oscillators, with coordinates ϕ_1, \dots, ϕ_L and conjugated momenta π_1, \dots, π_L :

$$\mathcal{H} = \frac{1}{2} \sum_{n=1}^L \pi_n^2 + \frac{1}{2} \sum_{n,n'=1}^L \phi_n K_{nn'} \phi_{n'}. \quad (42)$$

The ground state of the above Hamiltonian has the following form

$$\Psi_0 = \left(\frac{\det K^{1/2}}{\pi^L}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\langle\phi|K^{1/2}|\phi\rangle}. \quad (43)$$

One can calculate the two point correlators $(X_D)_{ij} = \text{Tr}(\rho_D \phi_i \phi_j) = (K^{-\frac{1}{2}})_{ij}$ and $(P_D)_{ij} = \text{Tr}(\rho_D \pi_i \pi_j) = (K^{\frac{1}{2}})_{ij}$ using the K matrix, where ρ_D is the reduced density matrix of the domain D . The square root of this matrix, as well as its inverse, can be split up into coordinates of the subsystems D and \bar{D} , i. e.,

$$K^{-1/2} = \begin{pmatrix} X_D & X_{D\bar{D}} \\ X_{D\bar{D}}^T & X_{\bar{D}} \end{pmatrix}, \quad K^{1/2} = \begin{pmatrix} P_D & P_{D\bar{D}} \\ P_{D\bar{D}}^T & P_{\bar{D}} \end{pmatrix}$$

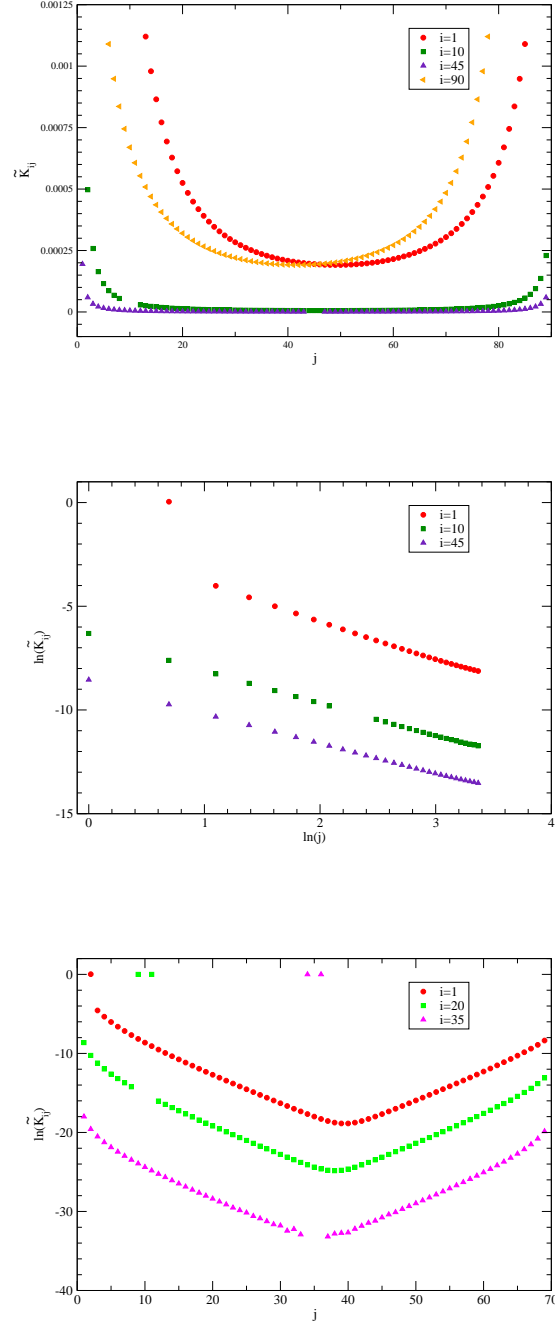


Figure 2. (Color online) Top: \tilde{K}_{ij} with respect to j after performing measurements on 10 contiguous sites for discrete Laplacian with $L = 100$. Middle: The same quantity in the log-log plot up to $j = 30$. Bottom: $\ln \tilde{K}_{ij}$ with respect to j for the massive case with $m = 0.3$ and $L = 80$ and after performing measurements on 10 contiguous sites.

The spectra of the matrix $2C = \sqrt{X_D P_D}$, can be used to calculate the von Neumann

and Rényi entanglement entropies,

$$S = \text{tr} \left[\left(C + \frac{1}{2} \right) \ln \left(C + \frac{1}{2} \right) - \left(C - \frac{1}{2} \right) \ln \left(C - \frac{1}{2} \right) \right], \quad (44)$$

$$S_\alpha = \frac{1}{\alpha - 1} \text{tr} \left[\ln \left(\left(C + \frac{1}{2} \right)^\alpha - \left(C - \frac{1}{2} \right)^\alpha \right) \right]. \quad (45)$$

Now if we make a measurement on the position of all the oscillators $\{\phi_i\} \in A$ they will take some definite values and will get decoupled from the rest of the oscillators. In other words, the post-measurement state will be the same as (43) but instead of $K^{1/2}$ we need to consider $(K^{1/2})_{\bar{A}}$ which is a subblock of the matrix $K^{1/2}$ corresponding to the oscillators in the subsystem \bar{A} . This means that we now have a new Gaussian wave function and one can calculate its bipartite entanglement entropy with the formulas (44) and (45). There is a simple interpretation for the wave function after measurement. It is just the ground state of the Hamiltonian (42) with $K = -\tilde{K} = ((K^{1/2})_{\bar{A}})^2$. It is not difficult to see that this Hamiltonian is highly non-local, for example, if we start with a discrete Laplacian in one dimension $K_{ij} = -\delta_{i,j-1} - \delta_{i,j+1} + 2\delta_{i,j}$ the \tilde{K} will have the form shown in Figure 2. Note that the \tilde{K}_{ii} is usually a number very close to -2 and $\tilde{K}_{i,i+1} = \tilde{K}_{i+1,i}$ is very close to 1 and the rest of the couplings are much smaller than these three couplings. As it is shown in figure 2 for site $i = 1$ the couplings decrease like a power-law but after reaching the middle of the original system they start to increase. It seems that this behavior is generic for all the oscillators up to the middle of the system. The sites beyond the middle of the system just show reverse behavior. The behavior in the massive case $K_{ij} = -\delta_{i,j-1} - \delta_{i,j+1} + (2 + m^2)\delta_{i,j}$ is a bit different. In this case $\tilde{K}_{i,j}$ decreases exponentially as it is shown in Figure 2. In this case the only significant elements of the matrix \tilde{K} are $\tilde{K}_{i,i+1} = \tilde{K}_{i+1,i} = 1$ and $\tilde{K}_{ii} = -2 - m^2$ which means that \tilde{K} is actually almost identical to the K matrix of the local massive Laplacian. This result will have significant consequences in the next sections when we discuss post measurement entanglement entropy of massive systems. It is worth mentioning that we expect that the post-measurement wave function is the ground state of a non-local Hamiltonian in generic systems. This makes the study of entanglement entropy in non-local Hamiltonians interesting also from this perspective, for a list of studies on the entanglement entropy of non-local systems see [27, 28, 29, 30]. In the upcoming sections, we use the above two versions of couplings (massless and massive) as the discretized versions of our free bosonic field theory.

5.1. Numerical results for disconnected regions

In this subsection we check the validity of the results presented in the section 3 by numerical means. The equations (20) and (36) are the classical results that have been checked before by many methods [2]. The equation (38) is already checked numerically in [7]. Here we study the entanglement entropy of two regions completely decoupled from each other, in other words, we check the equations (28) and (40). In $1 + 1$ dimensions the smallest scaling dimension in the Klein-Gordon field theory is $\Delta = 1$ which is the one corresponds to the field $\partial\phi$. Consequently we expect that the entanglement entropy

decays like a power-law $(\frac{l}{s})^{\Delta(\alpha)}$ with an exponent which is α dependent up to $\alpha = 1$ and then it saturates, see equation (28). In other words the exponent will be

$$\Delta(\alpha) = \begin{cases} 2\alpha, & \alpha < 1 \\ 2 & \alpha \geq 1, \end{cases} \quad (46)$$

The numerical results depicted in the figures 3 indeed confirm our expectations. We also checked the formula (40). For the Klein-Gordon field theory we should have power-law decay $(\frac{l}{L})^{\tilde{\Delta}(\alpha)}$ with

$$\tilde{\Delta}(\alpha) = \begin{cases} 4\alpha, & \alpha < 1 \\ 4 & \alpha \geq 1, \end{cases} \quad (47)$$

The results shown in figure 4 is in a good agreement with the above formula.

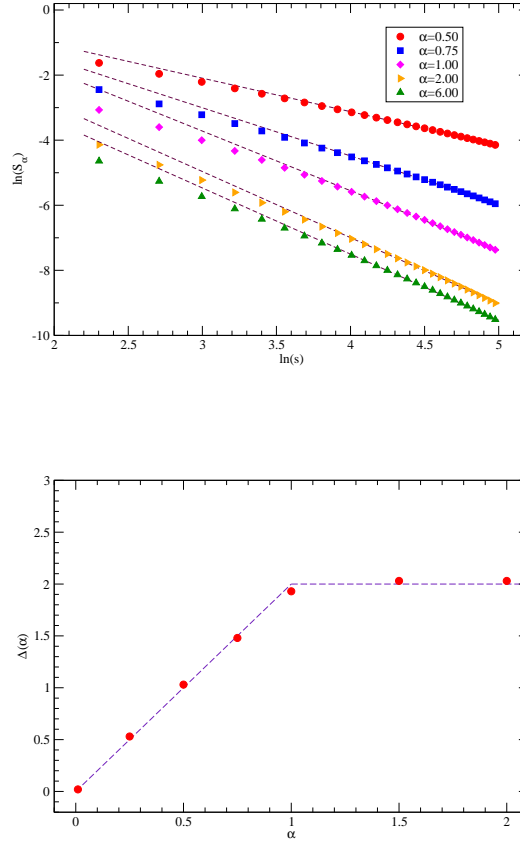


Figure 3. (Color online) Top: log-log plot of the Rényi entropy for the two disconnected regions. From top to bottom the full lines correspond to equation (28) with $\Delta(\alpha)$ coming from the equation (46). Down: the exponent of the power-law i.e. $\Delta(\alpha)$, with respect to α . The size of the total periodic system $L = 1000$ and the size of the region B is $l = 10$. The exponents are derived by fitting the data in the region $s \in (80, 150)$. The full line is the analytical result (46).

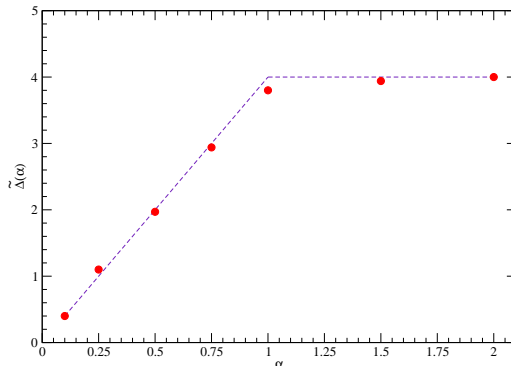


Figure 4. (Color online) the exponent of the power-law i.e. $\tilde{\Delta}(\alpha)$, with respect to α . The size of the total periodic system $L = 600$ and the size of the regions B and \bar{B} is taken $l \in (20, 50)$. The full line is the analytical result (47).

5.2. 1 + 1 dimensional massive QFT

In this subsection we would like to investigate bipartite entanglement entropy after partial measurement in 1 + 1 dimensional massive Klein-Gordon field theory.

When the field theory is massive the bipartite entanglement entropy follows the area-law [26, 2]. There are also many concrete proofs regarding the validity of the area-law in the 1 + 1 dimensional gapped systems, see [31, 32, 33, 34]. In massive local quantum field theories, it is shown [2] that the entanglement entropy of the system is given by

$$S_\alpha = -\kappa \frac{c}{12} \left(1 + \frac{1}{\alpha}\right) \ln m + \dots, \quad (48)$$

where c is the central charge of the system which is equal to 1 for Klein-Gordon field theory, m is the mass and $\kappa > 0$ is the number of contact points between two subsystems. We calculated the same quantity after partial measurement, as we discussed before, for different cases. Interestingly we found that the equation (48) is valid also after we decouple the region A from the system. One just needs to consider κ as the number of contact points between B and \bar{B} . We checked this for periodic boundary conditions for the cases that B and \bar{B} have one and two contact points. The results for $\alpha = 1$ are shown in the Figure 5. Similar results are valid also for $\alpha > 1$ confirming the equation (48). The above results look consistent with this picture that the entanglement between two regions is related to the couplings just around the interfaces of the two regions. Since as we discussed before the local measurements in parts of the system which is far from the boundary between B and \bar{B} disturb very little the couplings around the interfaces one might find it natural to have still the area-law after the partial measurement. Although post measurement area-law is probably a generic behavior of gaped systems the equation (48) is valid only in those cases in which we make our measurements in the conformal basis. We notice here that since in our system the correlations $\text{Tr}(\rho_D \pi_i \pi_j)$

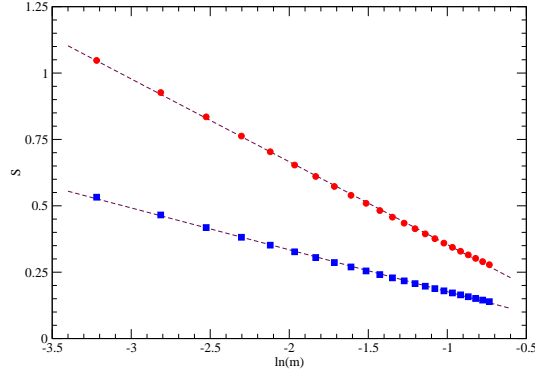


Figure 5. (Color online) Post measurement von Neumann entanglement entropy in massive field theory in $1 + 1$ dimension. The upper data correspond to when B and \bar{B} have two common boundary points and the lower data correspond to when B and \bar{B} have one common boundary. The full lines are the equation (48) with the coefficient of the logarithm from top to bottom being equal to -0.31 and -0.16 respectively. Mass range is chosen in a way that $1 < \frac{1}{m} < L$, where in the above L the size of the whole system is 100.

and $\text{Tr}(\rho_D \phi_i \phi_j)$ decay exponentially even after the measurement, based on [33], one naturally expects the area-law. This might be a good starting point for an analytical approach to the problem in the most general basis.

We also calculated the entanglement entropy in the case $l \ll s_1 = s_2 = s$ and found that the entanglement entropy decays exponentially. In other words we have

$$S_\alpha \underset{s \rightarrow \infty}{\asymp} e^{-\gamma(\alpha)ms}, \quad (49)$$

where $\gamma(\alpha)$ is a number that decreases with increasing α .

6. Conclusions

In this paper we discussed the entanglement entropy of the ground state of the conformal field theories in the $1 + 1$ dimension after projective measurement in the conformal basis in part of the system. Using the boundary conformal field theory methods we first rederived the known results of [1] and [7]. Then we further showed that if the two regions that we would like to calculate their entanglement with respect to each other got spatially decoupled after the projective measurement the entanglement entropy decays like a power-law with respect to the distance with an exponent which is equal to $2\alpha\Delta$ for $\alpha \leq 1$ and 2Δ for $\alpha > 1$, where Δ is the smallest scaling dimension in the spectrum of the system. Similar calculations are also done in the presence of the finite size effects. We then checked our results using numerical calculation on the Klein-Gordon field theory. We also provided numerical results regarding the massive case. In particular, we showed that after the projective measurement the bipartite entanglement entropy of the unmeasured oscillators follows the area-law as far as the two parts are connected.

When due to the measurement region they are disconnected the entanglement entropy decays exponentially. It will be very interesting to check, analytically or numerically, the validity of our results in other 1+1 dimensional systems such as spin chains. Calculating the same quantities in higher dimensions is also stimulating. Finally, it will be nice to investigate the same kind of questions using holographic techniques [35].

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Appendix: Conformal maps

In this appendix we provide some details regarding the conformal maps used in the main text. The conformal map from a whole plane with two slits and a branch cut to annulus can be derived as follows: the first step is mapping the whole plane with two slits to a whole plane with two symmetric slits as it is shown in the Figure 1. This can be done with the map $\tilde{z}(z)$ which has the following form:

$$\begin{aligned}\tilde{z}(z) &= \frac{2a}{k} \frac{z}{bz+1} - \frac{1}{k}, \\ a &= \sqrt{\frac{s_2(s_2+l)}{s_1(s_1+l)}} \frac{1}{l+s_1+s_2}, \\ b &= \frac{\sqrt{s_1 s_2 (l+s_1)(l+s_2)} - s_2(l+s_1)}{(l+s_1)(s_1 s_2 - \sqrt{s_1 s_2 (l+s_1)(l+s_2)})},\end{aligned}\tag{A.1}$$

with the parameter k given by

$$k = 1 + 2 \frac{s_1 s_2 - \sqrt{s_1 s_2 (l+s_1)(l+s_2)}}{l(l+s_1+s_2)}.\tag{A.2}$$

Then the whole plane with the symmetric slits and a branch cut can be mapped to annulus with a branch cut by the following conformal map [36]:

$$w_1(\tilde{z}) = e^{-\frac{h}{2}} e^{h \frac{\text{sn}^{-1}(\tilde{z}, k^2)}{2\mathcal{K}(k^2)}},\tag{A.3}$$

$$h = 2\pi \frac{\mathcal{K}(k^2)}{\mathcal{K}(1-k^2)},\tag{A.4}$$

The final step is the uniformization of the domain by using the conformal map $w_\alpha(z) = w_1(z)^{\frac{1}{\alpha}}$.

The same procedure as above can be used also for the periodic system. We first use the conformal map $e^{2\pi i \frac{z}{L}}$ to map the cylinder with two slits and a branch cut to

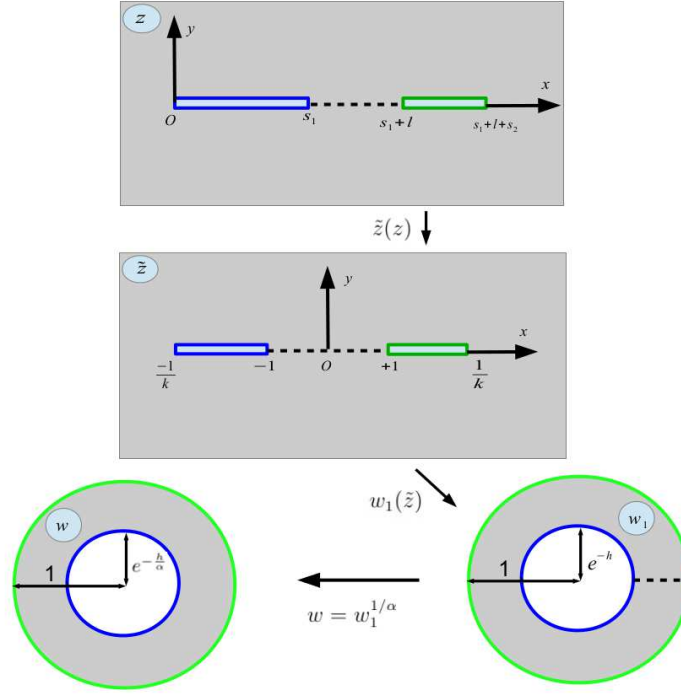


Figure A1. (Color online) Mapping between different regions. The whole plane with two slits (blue and green) and branch cut (dashed line) can be mapped to the whole plane with symmetric slits by the map \tilde{z} . Using $w_1(\tilde{z})$ we can map the whole plane with symmetric slits and a branch cut to annulus with a branch cut. Finally $w_1(z)^{\frac{1}{\alpha}}$ removes the branch cut.

the whole plane with slits and a branch cut. Then we map the remaining region to a whole plane with symmetric slits by using a mobius map. The above two steps can be summarized as the equation (32). Now the whole plane with symmetric slits and a branch cut can be mapped to an annulus by the map $\left(w_1(\tilde{z})\right)^{\frac{1}{\alpha}}$.

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